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Convex Functions, Harmonic Maps, and the Stability of Hamiltonian Systems

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Abstract: Trajectories of conservative dynamical systems are particular examples of harmonic maps. If Y is the configuration space of a dynamical system, then a trajectory of the system is a harmonic map from the real line into Y. More generally, let X and Y be riemannian manifolds with X compact. We show that the image of any harmonic map f from X to Y cannot be contained in domains which are too small; specifically, that the image of any such f cannot be contained on any domain which supports a convex function. From a modification of the proof we show that, except in the neighborhoods of certain exceptional points, a trajectory of a dynamical system cannot lie entirely in any such domain. This fact leads to criteria for the growth and instability of dynamical systems.

INTRODUCTION

In Part I of this report we announce some results which we have recently obtained in the theory of harmonic maps, of which a more detailed account with proofs will be given in a forthcoming paper [1]. Part II is devoted to a discussion of some possible applications of these results to the study of the stability of Hamiltonian systems.

For the convenience of those readers who may wish to go immediately to Part II we now introduce some necessary concepts. Let Y be a smooth riemannian manifold with local coordinates y^i, metric tensor h_{ij}, and Christoffel symbols Γ^k_{ij}. The coefficients of the second-order covariant differential of a function F with continuous second-order derivatives are given by

$$F_{ij} = \frac{\partial^2 F}{\partial y^i \partial y^j} - \Gamma^k_{ij} \frac{\partial F}{\partial y^k},$$

where here, as always, we use the tensor summation convention. We shall say that F is convex in a domain D of Y *iff* at every point p of D we have $F_{ij}\xi^i\xi^j > 0$ for every nonzero contravariant vector ξ. (In symbols, we shall write $(F_{ij}) > 0$ (in D).) A domain D will be said to be convex supporting *iff* there exists a convex function F on D. (A more general definition is used in Part I, section 1-A.) Examples of convex-supporting manifolds and domains are given in section 1-C of Part I, Theorem 3 in section 1-B, and remark 2 in section 1-A.

Let $y = y(t)$ be a curve on Y. The second-order covariant derivative is given by

$$\frac{D^2 y^i}{dt^2} = \frac{d^2 y^i}{dt^2} + \Gamma^i_{jk} \frac{dy^j}{dt} \frac{dy^k}{dt}.$$

Hence the analytic condition that the curve be a geodesic is given by $D^2y^i/dt^2 = 0$. Let F be a C^2 function, $y = y(t)$ be a curve on Y, and $G(t) = F[y(t)]$. Then $G'(t) = F_i\, dy^i/dt$ and $G''(t) = F_{ij}$ $(dy^i/dt)(dy^j/dt) + F_i(D^2y^i/dt^2)$. Hence, F is convex in a domain D *iff* for every geodesic $y = y(t)$ in D we have $d^2F[y(t)]/dt^2 > 0$.

In Part II we shall show how this notion of convexity leads to Lyapounov-like criteria for the stability of conservative dynamical systems.

NRL Problem B01-11; Project RR 003-02-41-6153. This is an interim report on one phase of the problem; work is continuing. Manuscript submitted June 19, 1970.

PART I
Harmonic Maps

1-A. PRELIMINARIES

X and Y will always denote C^∞ riemannian manifolds, and it will always be assumed that X is compact. Greek (Latin) letters will be used for objects attached to $X(Y)$. Local coordinates will be written $x = (x^\alpha)$, $y = (y^i)$, and the corresponding metric tensors, Christoffel symbols, and curvature tensors will be denoted by $g = (g_{\alpha\beta})$, $h = (h_{ij})$, $\Delta^\alpha_{\beta\gamma}$, Γ^i_{jk}, $R_{\alpha\beta\gamma\delta}$, S_{ijkt}. Indices of tensors will be raised and lowered in the usual fashion, and we shall always use the tensor summation convention.

Let f: $X \to Y$ be a C^2 map which is locally given by $y = y(x)$. We set

$$y^i_\alpha = \frac{\partial y^i}{\partial x^\alpha}\,;\; y^i_{\alpha\beta} = \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta} - \Delta^\gamma_{\alpha\beta} y^i_\gamma + \Gamma^i_{jk} y^j_\alpha y^k_\beta\,. \tag{1}$$

Then y^i_α and $y^i_{\alpha\beta}$ transform like tensors under coordinate transformations $x \to \bar{x}$, $y \to \bar{y}$. f is said to be harmonic if $g^{\alpha\beta} y^i_{\alpha\beta} = 0$, $(i = 1,2,\ldots,di(Y))$. For motivation and some results in the theory of harmonic map see Refs. [2-4]. We recall some basic definitions and results: f is said to be totally geodesic if f sends geodesics into geodesics or, equivalently, if f_{**}, the differential of the differential of f, sends the horizontal sub-bundle of $TT(X)$ into the horizontal sub-bundle of $TT(Y)$. The analytic condition that f be totally geodesic is given by $y^i_{\alpha\beta} = 0$. Thus, every totally geodesic map is harmonic. f is said to be an isometric map if $f^*g = h$; *i.e.*, $g_{\alpha\beta} = h_{ij} y^i_\alpha y^j_\beta$. An isometric map f: $X \to Y$ is totally geodesic (hence harmonic) provided $di(X) = di(Y)$. The composition of two totally geodesic maps is totally geodesic. The composition of a totally geodesic map with a harmonic map is harmonic according to the following scheme: totally geodesic $\circ$ harmonic $=$ harmonic. The composition of two harmonic maps is not necessarily harmonic.

A 1-form $\omega = (\omega_i)$ defined on a domain D of Y will be said to be convex if its covariant differential is a positive definite form; *i.e.*, if for every $p \in D$ and non-zero vector $\xi \in T_p(Y)$, we have $\omega_{ij}(p)\, \xi^i \xi^j > 0$, where $\omega_{ij} = \partial w_i/\partial y^j - \Gamma^k_{ij}\omega_k$. A function F will be said to be convex if dF is convex. We write $(\omega_{ij}) > 0$, $(F_{ij}) > 0$ to indicate that ω, F are convex. (The reason for using the strict inequality will be apparent from the proof of Theorem 1 below.) A manifold Y will be said to be convex supporting if every domain D and Y with compact closure supports a convex form. We note the following:

1. A compact manifold Y cannot be convex supporting.

Proof. Let ω be any 1-form on Y. We can assume that Y is orientable, since otherwise we could apply the following argument to the lift of ω to the two-leaved orientable cover of Y. It is well-known that

$$\int_Y h^{ij}\omega_{ij}\sqrt{h}\; dy = 0.$$

Therefore, ω cannot be convex, since this would imply that the integrand is positive.

2. Every point has a convex-supporting neighborhood. For in a coordinate neighborhood of the point define

$$F(y) = \frac{1}{2}\sum\, (y^i)^2.$$

Then

$$F_{ij} = \delta_{ij} - \sum_k \Gamma^k_{ij} y^k.$$

Hence, from the continuity of Γ^k_{ij}, $(F_{ij}) > 0$ in some neighborhood of the point.

1-B. STATEMENT OF RESULTS

We shall write Rie $(Y) \leqslant 0$ to indicate that the riemannian sectional curvatures are all non-positive, and given two 2-forms ω, θ we write $(\omega_{ij}) > (\theta_{ij})$ to indicate that $(\omega_{ij} - \theta_{ij}) > 0$. We use Eisenhart's conventions concerning the curvature tensors, so that, in particular, the Ricci curvature is positive if $(-R_{ij}) > 0$ [5]. We now present our results.

THEOREM 1. *If Y is convex supporting, then every harmonic map from X to Y is necessarily constant.*

THEOREM 2. *Suppose $\pi_1(X)$ is finite, and that Y has a convex-supporting covering space; i.e., we suppose there exists a covering $\tilde{Y} \to Y$ for which $\tilde{Y}$ is convex supporting with respect to the lifted metric of Y. Then every harmonic map from X to Y is constant.*

Remarks. We recall that $\pi_1(X)$ is finite if the universal covering space of X is compact. Also, we may as well assume that the covering $\tilde{Y} \to Y$ is universal, since if it is not, the universal covering $U \to Y$ can be factored through the given covering, and the induced covering $U \to \tilde{Y}$ can be used to lift the convex form on Y to a convex form on U.

THEOREM 3. *Suppose Y is complete, simply connected, and that Rie $(Y) \leqslant 0$. Then Y is convex supporting. In particular, there exists a convex function on Y.*

Remark. None of the conditions of this theorem are necessary. In particular, there exist convex-supporting manifolds Y with Rie $(Y) \geqslant 0$ (see section 1-C below).

Before going on to the next theorem, we mention some corollaries. First, note that the universal covering space of any complete riemannian manifold with non-positive sectional curvatures satisfies the conditions of Theorem 3. Hence, if $\pi_1(X)$ is finite and Y is a complete manifold with Rie $(Y) \leqslant 0$, then every harmonic map from X to Y is constant.

If the Ricci curvature of X satisfies $(-R_{\alpha\beta}) > \lambda(g_{\alpha\beta})$ for some positive constant λ, then according to a well-known theorem of Myers, $\pi_1(X)$ is finite (see Ref. [6], p. 105), so that the only harmonic maps from X to a space with a convex-supporting covering are constants. This condition on the Ricci curvature holds if X, say, is S^n or $\mathbb{P}^n$, $n \geqslant 2$, endowed with the standard riemannian structures, or if X is a Lie group with trivial center. However, it should be noted that if the standard metric of any of these manifolds X is replaced with an arbitrary metric, then it still remains true that every harmonic map from X to a manifold with convex-supporting cover is necessarily constant.

Remark. These results should be compared with the Corollary on page 124 of Ref. [2] and also with Ref. [4]. For example, Eells and Sampson show that if Rie $(Y) \leqslant 0$ and if the Ricci curvature of X is non-negative and positive at least at one point, then every harmonic map from X to Y is constant.

Before stating our next theorem we recall that a map F is said to be proper if F^{-1} (compact) = compact.

THEOREM 4. *Let F be a convex C^2 function on a complete manifold Y. Then either F has no critical points or F has a unique critical point p_0. In the latter case F is proper, $F(p) > F(p_0)$ for all other points of p of Y, and Y is contractible.*

COROLLARY. *Let F and Y be as above and suppose that $(F_{ij}) > \lambda(h_{ij})$ for some positive constant λ. Then F has a unique critical point and the conclusions of the theorem follow.*

1-C. EXAMPLES OF CONVEX-SUPPORTING DOMAINS AND MANIFOLDS

1. In E^3 the surface (paraboloid of revolution) defined by $x^2 + y^2 = z^2$ is convex supporting, with convex function $F = x^2 + y^2$. Note that the riemannian curvature of this manifold is positive.

2. The surface $x^2 + y^2 = e^{-z}$ is convex supporting, with $F = x^2 + y^2$. The curvature of this manifold is negative. Hence, neither the curvature nor connectivity assumption of Theorem 3 is necessary for a manifold to be convex supporting.

3. A closed geodesic on Y is a harmonic map from the circle to Y. Hence, the hyperboloid $x^2 + y^2 = 1 + z^2$ is not convex supporting because it contains a closed geodesic at $z = 0$. However, the upper and lower halves of the hyperboloid are convex supporting, again with $F = x^2 + y^2$. Note that the manifolds in remarks 2 and 3 are homeomorphic and have negative riemannian curvature.

4. Let S^n be the n-sphere endowed with the usual metric and let S^n_+ be the hemisphere containing a fixed pole p. We show that S^n_+ is a maximal open convex-supporting domain: Let $F(q) = r^2(q)$, where $r(q)$ = polar distance pq. A direct calculation shows that F is convex, so that S^n_+ is convex supporting. Next, suppose that F is any convex function on S^n_+ which extends to a convex function F on an open domain D which contains S^n_+. D cannot contain the equator S^{n-1}, since if it did, D would contain a closed geodesic. Let $0(n)$ be the orthogonal group identified with the group of isometries $S^n \to S^n$, which leave p fixed. Then for $\phi \in 0(n)$, $F \circ \phi$ is convex. Letting $0(n)$ operate from the left on S^n_+, we set

$$F_1(q) = \int_{0(n)} F(gq)\, d\mu(g),$$

where μ is Haar measure on $0(n)$. It is easy to show that F_1 is a convex function which is invariant under $0(n)$ and extends to a convex function on an open domain D_1 which contains D and also the equator S^{n-1}. But this is impossible. Therefore, S^n_+ is a maximal open domain which supports a convex function. Replacing F by a convex 1-form ω in the argument, we use a similar construction with

$$\omega_1(q) = \int_{0(n)} g^* \omega(gq)\, d\mu(g),$$

where g^* is the map induced on $T^*(Y)$ by the map $q \to gq$.

A similar argument shows that the upper and lower halves of the hyperboloid in remark 3 are maximal open convex-supporting domains.

5. Maximal convex-supporting domains need not exist. For example, let $\{I_n\}$ be a sequence of proper closed segments of nonzero length on the unit circle C with $I_{n+1} \subset I_n$ and such that $\cap I_n$ = point. The complements $\{I_n^*\}$ are open and satisfy $I_n^* \subset I_{n+1}^*$. It is easy to show that each I_n^* is convex supporting, but $\cup I_n^* = C - \{\text{point}\}$ is not convex supporting. Hence, it follows that C contains no maximal convex-supporting domains.

PART II

Applications to the Stability of Hamiltonian Systems

1-A. GEODESIC FLOWS

We recall that a riemannian manifold is said to be complete *iff* every geodesic can be indefinitely continued, and that this condition holds *iff* the manifold is complete with respect to the metric induced on the manifold by the riemannian metric.

THEOREM. *Let M be a complete riemannian manifold and let D be a compact domain of M which supports a convex function F. Then any geodesic which enters D must leave D.*

Such a geodesic may course return to D, but D cannot contain any entire geodesic $\gamma(t)$ $(-\infty < t < \infty)$ or any half-ray $\gamma(t)$ $(t \geqslant t_0$, for some $t_0)$.)

Proof. Let $\gamma = \gamma(t)$ be a geodesic for which $\gamma(t_0) \in D$, and suppose that $\gamma(t)$ remains in D for all $t \geqslant t_0$. Let $G(t) = F[\gamma(t)]$ and let ξ denote the velocity vector of γ. As shown in the Introduction,

$$G' = F_i \xi^i \quad \text{and} \quad G'' = F_{ij} \xi^i \xi^j .$$

Since D is compact, there exists a positive constant λ such that the bilinear form $(F_{ij} - \lambda g_{ij})$ is positive definite (in D). And since the velocity vector along a geodesic has constant length $(g_{ij}\xi^i\xi^j = 1)$, we have $G''(t) \geqslant \lambda$, $t \geqslant t_0$. Hence, $G(t)$ increases without bound as $t \to \infty$. But this is impossible since F must be bounded on the compact domain D.

1-B. THE JACOBI METRIC

Let V be a continuously differentiable function on a manifold Y. The trajectories of the conservative dynamical system arising from the "potential" V are the solutions to the dynamical equation $D^2y^i/dt^2 = -\, h^{ij}(\partial V/\partial y^j)$. In the special case $V = 0$, the trajectories are merely the geodesics of the manifold.

We recall some basic facts about the Jacobi metric [7]. Let H be a constant. The Jacobi metric $\hat{h}_{ij}$ is defined by

$$\hat{h}_{ij} = 2(H - V)h_{ij} \tag{1}$$

at points where $H - V > 0$. It is well known that the geodesics corresponding to this metric are (re-parametrized) trajectories of the dynamical system with potential V and total energy H. Hence, a discussion of Hamiltonian systems can be reduced to the case of geodesic flows.

For later reference we note that the Christoffel symbols $\hat{\Gamma}^i_{jk}$ associated with $\hat{h}_{ij}$ are given by

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} - \frac{1}{2(H - V)}\left(\delta^i_j \frac{\partial V}{\partial y^k} + \delta^i_k \frac{\partial V}{\partial y^j} - h^{ir} h_{jk} \frac{\partial V}{\partial y^r}\right). \tag{2}$$

1-C. GENERAL REMARKS ON THE USE OF THE JACOBI METRIC

1. It should be emphasized that the Jacobi metric is constructed for each fixed value of H. A given function F defined on a fixed domain D may be convex with respect to the Jacobi metric corresponding to some values of H but not for others.

2. The Jacobi metric degenerates at those points for which $H \geqslant V$. In particular, a trajectory can converge to a point at which $H = V$. (See Example 2 below.)

3. Suppose D is a maximal *open* convex-supporting domain, with compact closure, which supports a function F which is convex in D and continuous on the closure of D. It is conceivable that a geodesic which enters D might remain in D (so that it would approach the boundary of D). Referring to the proof of the theorem, to exclude this possibility it suffices to show that $G'(t) \geqslant 0$ for some $t \geqslant t_0$. On the other hand, if such a geodesic remains in D, we must have $G' < 0$, and $G(t)$ must converge to its infimum as $t \to \infty$.

2. EXAMPLES

We give two examples to illustrate the basic calculations involved in these convexity methods.

Example 1. The Kepler Problem: Let $\{x_i\}$ denote standard coordinates and $\langle\, , \rangle$ the standard inner product on euclidean n-space. Set $r^2 = \langle x,x \rangle = \Sigma x_i^2$. The potential function for the Kepler problem is given by

$$V = -\frac{C}{r}\,, \text{ so that } V_i = \frac{\partial V}{\partial x_i} = \frac{Cx_i}{r^3}\,. \tag{3}$$

Set

$$F(x) = \frac{1}{2} r^2, \text{ so that } \frac{\partial F}{\partial x_i} = x_i, \ \frac{\partial^2 F}{\partial x_i \partial x_j} = \delta_{ij}. \tag{4}$$

We have $h_{ij} = \delta_{ij}$, $\Gamma^i_{jk} = 0$. Hence, from (1) and (2),

$$\hat{h}_{ij} = 2(H - V)\delta_{ij} \tag{5}$$

and

$$\hat{\Gamma}^i_{jk} = -\frac{C}{2r^3(H-V)}\,(\delta^i_j x_k + \delta^i_k x_j - \delta_{jk} x_i)\,. \tag{6}$$

From the relation

$$F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} - \hat{\Gamma}^k_{ij}\,\frac{\partial F}{\partial x^k}$$

we get

$$F_{ij} = \left(1 - \frac{C/r}{2(H-V)}\right)\delta_{ij} + \left(\frac{C}{r^3(H-V)}\right)x_i x_j = a\delta_{ij} + b x_i x_j,$$

where $a,b = etc.$

We wish to determine when $(F_{ij}) > 0$; since $F_{ij} = F_{ji}$, this reduces to determining the eigenvalues of the matrix M given $M_{ij} = F_{ij} = a\delta_{ij} + b x_i x_j$.

Case 1. $H \geqslant 0$, so that $H - V = H + C/r > 0$.

Let ξ be an eigenvector with eigenvalue μ. Then $M\xi = \mu\xi$ is equivalent to

$$a\xi + b\,\langle x,\xi \rangle x = \mu\xi\,.$$

There are $(n-1)$ independent vectors ξ with $\langle x,\xi\rangle = 0$. Hence, a is an eigenvalue of multiplicity $(n-1)$. Also,

$$a = 1 - \frac{C/r}{2(H + C/r)} > \frac{1}{2}. \tag{7}$$

The remaining solution is given by $\xi = x$, in which case (8) reduces to $ax + br^2x = x$, so that, taking the inner product of both sides with x, we get

$$\mu = a + br^2 = 1 + \frac{C/r}{2(H + C/r)} > \frac{3}{2}. \tag{8}$$

Hence, if $H > 0$, the entire space is convex supporting with convex function F. This means that no trajectory $x = x(t)$ can remain in any compact set for all t greater than any fixed value. In fact, suppose $r = r(t)$ is bounded below. Then the solution is defined for all $t \geqslant 0$, and from convexity, $G(t) = F[x(t)]$ must eventually become monotonic; *i.e.*, we have proved the elementary fact that if $H > 0$ and $r = r(t)$ is bounded below, then $r(t) \to \infty$ as $t \to \infty$.

Case 2. $H < 0$. In this case the domain D given by $r < -C/H$ is convex supporting, and the analysis shows that every trajectory which enters D must leave D. (Of course, such trajectories are ellipses, so that the trajectories return to D.)

Example 2. We consider the motion of a particle constrained to move on an ordinary 2-sphere in the presence of a uniform gravitational field.

Let the embedding of the sphere into euclidean 3-space be given by

$$w = w(\theta,\varphi) = (\sin\theta\cos\varphi)\, e_1 + (\sin\theta\sin\varphi)\, e_2 + (\cos\theta)\, e_3,$$

where e_1, e_2, e_3 are the standard basis vectors in 3-space, θ = colatitude, φ = longitude. The potential is given by $V = \cos\theta$. The force field is directed downward from the north pole ($\theta = 0$) to the south pole ($\theta = \pi$).

Fix a value of H, ($H > -1$). Using θ,φ as generalized coordinates, it is easy to show that a trajectory with total energy H is given by the circular orbit

$$\theta = \theta_o, \tag{10}$$

where

$$\frac{1}{|\cos\theta_o|} - 3|\cos\theta_o| = 2H$$

and

$$\dot{\varphi}^2 = -1/\cos\theta_o. \tag{11}$$

Note that as a consequence of (11) this orbit occurs on the southern hemisphere ($\theta > \pi/2$). Let D be the domain given by

$$\frac{1}{|\cos\theta|} - 3|\cos\theta| < 2H,\ \theta > \pi/2.$$

Presently we shall show that D is convex supporting. We ask whether there exists a trajectory in D (with energy H) which winds up toward the boundary of D, $\theta = \theta_o$. The answer is "no." Hence, any trajectory with total energy H which enters D must leave D.

Proof. Let $F = F(\theta,\varphi) = 1/2\ \langle w - e_3, w - e_3\rangle$. (Note that $f = 1 - V$.) Direct calculation gives

$$(F_{ij}) = \begin{bmatrix} \cos\theta - \dfrac{\sin^2\theta}{2(H-\cos\theta)} & 0 \\ 0 & \left(\cos\theta + \dfrac{\sin^2\theta}{2(H-\cos\theta)}\right)\sin^2\theta \end{bmatrix}.$$

There is some difficulty about the fact that the spherical coordinate system is singular at the poles, but it is easy to show that D is convex supporing; *viz.*, $(-F_{ij}) > 0$ in D.

To prove that the non-existence of winding orbits, it suffices to show that $d(-F)/dt > 0$, if $\dot\theta < 0$. But $dF/dt = \langle dw/dt,\, e_3\rangle = d\langle w, e_3\rangle/dt = d\cos\theta/dt = (-\sin\theta)\dot\theta$.

Finally, to obtain an example of a trajectory converging to a point at which the Jacobi metric degenerates, consider the case of a particle initially at rest at the south pole which is given just enough impetus to move it up to the north pole. We have $H = 1$, $H - V = 0$ at $\theta = 0$. The equations of motion are easily solved, and one finds that $\theta \sim e^{-t}$ as $t \to \infty$.

ADDENDUM

I have recently learned of a paper by Bishop and O'Neill [8] in which the authors establish several implications of the existence of globally defined convex functions on a manifold (especially a manifold of non-positive curvature) for the structure of the manifold. Thus, they obtained Theorems 3 and 4 of this report, and much more besides.

REFERENCES

1. W.B. Gordon, "Convex Functions and Harmonic Maps," to be published
2. J. Eells, Jr., and J.H. Sampson, "Harmonic Mappings of Riemannian Manifolds," *Am. J. Math* **86**:109-160 (1964)
3. F.B. Fuller, "Harmonic Mappings," *Pro. Nat. Acad. Sci.* **40**:987-991 (1954)
4. P. Hartman, "On Homotopic Harmonic Maps," *Can. J. Math.* **19**:673-687 (1967)
5. L.P. Eisenhart, "Riemannian Geometry," Princeton:Princeton University Press, 1926
6. J. Milnor, "Morse Theory," *Annals of Math. Studies 51*, Princeton:Princeton University Press, 1963
7. D. Langwitz, "Differential and Riemannian Geometry," New York:Academic Press, p. 172, 1965
8. R.L. Bishop and B. O'Neill, "Manifolds of Negative Curvature," *Trans. A.M.S.* **145**:1-49 (1969)

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